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LONG-PERIOD EFFECTS IN NEARLY COMMENSURABLE CASES OF THE RESTRICTED THREE-BODY PROBLEM

by

Joachim Schubart

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Table of Contents

1. Introduction	1
2. Formulas	3
3. The long-period effects	7
A. The planar case	7
B. Nonplanar cases	10
C. The effects as a function of time	10
4. Application to real cases	13
A. The gaps	13
B. The groups	15
C. The outer commensurability cases	18
5. Acknowledgments	18
References	19
Appendix 1 : Canonical variables	22
Appendix 2 : Figures	23

LONG-PERIOD EFFECTS IN NEARLY COMMENSURABLE CASES OF THE RESTRICTED THREE-BODY PROBLEM[1]

by

Joachim Schubart[2]

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The long-period effects in nearly commensurable cases of the restricted three-body problem were studied according to the ideas of Poincaré. The secular and critical terms of the disturbing function were isolated by a numerical averaging process, by use of an IBM-7094 computer. Stability is found for all real asteroid orbits corresponding to the cases treated here. Periods of the variations are given for some forms of planar motion. Planar and nonplanar periodic solutions are indicated by the theory. No evidence is found for a disintegration of the Trojan and Hilda groups of asteroids. Brouwer's explanation of differences between the Hecuba and Hilda commensurabilities is supported. Author

1. Introduction

Minor planets with mean motions that are nearly commensurable with the mean motion of Jupiter offer a special problem within the general three-body problem. Some groups of minor planets, such as the Trojan and the Hilda groups, have mean motions that cluster around a commensurable value, while other commensurabilities seem to be avoided, as in the Hecuba or Hestia gaps in the distribution of the mean motions. Even when a gap exists, some minor planets have mean motions close to the commensurable value. The name-giving planets Hecuba and Hestia are examples. The mean motion of Griqua passed the exact value of the Hecuba commensurability in the year 1943 (Rabe, 1959). Hagihara (1961) completed a long list of papers containing the most important work relating to commensurability cases. Two recent publications of Rabe (1961, 1962) on periodic Trojan orbits of long period and an article by Brouwer (1963) with a theoretical explanation of the gaps should be added to the list.

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The restricted three-body problem was taken as the basis of the work done here. Therefore the mass of the minor planet is neglected and Jupiter is assumed to move around the sun on a circular orbit. The gravitational constant, the mass of the sun, and the distance between the sun and Jupiter were taken equal to unity. The mass of Jupiter is treated as a small quantity. For the numerical computations the value 1/1047.355 was ordinarily used, but the theory is developed with an optional value m for the mass of the disturbing body. It could also be used for two satellites of a planet if one of these is much smaller than the other.

It is assumed that the ratio of the mean motion of the minor planet to that of Jupiter is close to the value $(p + q)/p$, where p and q are relative prime integers $(p > 0)$. Small values of p and q give us the most important commensurabilities. Thus the ratios 1/1 and 3/2 belong to the Trojan and Hilda groups, and 2/1 and 3/1 to the Hecuba and Hestia gaps. For these and similar commensurabilities, the usual secular perturbation method does not hold, because one of the angular arguments in the series development of the disturbing function becomes critical. This means that the terms with this argument and its multiples give rise to long-period effects of predominant influence, which must be considered in a long-range perturbation theory. So far, no general secular theory is known that treats all or several commensurabilities at once. But methods for a single commensurability case are available. Important work in this field was done at the beginning of this century. Hill (1900) applied Delaunay's method to minor planet orbits of the Hecuba type. The secular terms and the terms with the critical argument are selected in the series development of the disturbing function. Poincaré (1902a) suggested a special system of canonical variables and a simple method for commensurability cases, which Andoyer (1903) applied to the Hecuba case. Curves were published which show the essential features of all possible forms of motion in a plane.

When Schwarzschild (1903) worked on the periodic orbits of the Hecuba case and their stability, he used the method of averaging to isolate the influence of the critical argument. The value of the considered function is computed for equidistant points in the periodicity interval of a short-period argument, while the critical and other long-period arguments remain at a constant value. The mean value then gives the isolated influence of the long-period arguments. No series development is necessary. A tabulation of the function for different values of the long-period variables is possible. This method is analogous to the Gaussian method of computing the secular perturbations in the general case of planetary motion (Hill, 1882), which now can be used to get the long-range effects in the motion of celestial bodies by electronic computers, if no commensurability exists (Musen, 1963). Moiseev (1945) has published a compilation of the Gaussian, the method described here, and other averaging methods.

In the present paper Poincaré's variables and method are used, but the averaging process was introduced to get the interesting part of the disturbing function for a selection of commensurability cases. With the aid of an IBM-7094 computer the averaged disturbing function could easily be tabulated for a large set of values of the remaining variables. No restriction in the eccentricity (e) is required. Even the case e = 1 can be treated, since the transformation to the regularizing eccentric anomaly is made in the formulas. Andoyer's curves were repeated and corrected for the larger e values. The corresponding curves were found for other commensurabilities. Nonplanar periodic solutions are indicated by the computations. The periods for the long-range variations in the Hecuba and Hilda cases were obtained for a number of orbits. Comparison with direct numerical integration proves this approximate-commensurability theory. It fails only in cases of close approach to Jupiter, or if the remainder part of the disturbing function must be supposed to be very important. For the Hilda group of asteroids no disintegration of the objects occurs, but a quite regular oscillation of long period will take place in the mean motions over the millennia.

2. Formulas

The canonical system of variables published by Poincaré (1902a) is adapted to the commensurability ratio (p + 1)/p, but can easily be generalized to the ratio (p + q)/p if Poincaré's integer n is changed to the rational number p/q. q may not be zero, but the Trojan case also can be treated with these equations, as is shown later. The generalized system is given by the equations

$$\frac{dU}{dt} = \frac{\partial F}{\partial \lambda}, \qquad \frac{dS}{dt} = \frac{\partial F}{\partial \sigma}, \qquad \frac{dT}{dt} = \frac{\partial F}{\partial \tau},$$

$$\frac{d\lambda}{dt} = -\frac{\partial F}{\partial U}, \qquad \frac{d\sigma}{dt} = -\frac{\partial F}{\partial S}, \qquad \frac{d\tau}{dt} = -\frac{\partial F}{\partial T},$$

$$F = n_1\left[U - (S + T)(p + q)/q\right] + \frac{1}{2}\left[U - (S + T)p/q\right]^{-2} + m\left[\Delta^{-1} - \zeta\right].$$

Here the unit of the time t is fixed by the choice already made for the other units; m is the mass of the disturbing body, and the equation $n_1 = \sqrt{1 + m}$ represents its mean motion; Δ^{-1} is the reciprocal distance of the minor planet from Jupiter, and ζ is the indirect term of the disturbing function and is given by the scalar product of the vectors from the sun to the two other bodies. If the osculating elements of the minor planet orbit referred to the sun and to the orbital plane of Jupiter are given, Poincaré's variables are obtained from the expressions

$$U = [(p + q)\sqrt{a} - p\sqrt{a}\cos\varphi\cos i]/q ,$$

$$S = \sqrt{a}\,(1 - \cos\varphi) ,$$

$$T = \sqrt{a}\cos\varphi\,(1 - \cos i) ,$$

$$\lambda = \ell - \ell_1,$$

$$\sigma = M - (\ell - \ell_1)(p + q)/q ,$$

$$\tau = \ell - \Omega - (\ell - \ell_1)(p + q)/q ,$$

where a is the semi-major axis; e is the eccentricity; φ is the eccentricity angle, $e = \sin\varphi$; i is the inclination; Ω is the longitude of the ascending node; M is the mean anomaly; ℓ is the mean longitude; and ℓ_1 is the mean longitude of Jupiter. For $i = 0$ one has $T = 0$, while τ becomes unnecessary. S vanishes with e. For small values of S and T, Poincaré introduces the variables

$$x = \sqrt{2S}\cos\sigma , \qquad y = \sqrt{2S}\sin\sigma ,$$

or

$$\xi = \sqrt{2T}\cos\tau , \qquad \eta = \sqrt{2T}\sin\tau .$$

The pair x,y can replace S,σ and ξ,η can replace T,τ in the canonical system of differential equations. The function F is regular in the variables $x,y,\ \xi,\eta$ if they are small.

T also vanishes with cos φ . This is irrelevant in the case of planar motion, as τ,T are unnecessary. In the general case, however, the inclination and the mean longitude become indeterminate. Variables describing the direction of the line of apsides are useful,then, together with the mean anomaly and other quantities, which are proportional to cos φ and give the orientation of the orbital plane. Several systems of canonical variables were found, which are adapted to commensurability cases (see Appendix 1).

During the averaging process fixed values are prescribed for U, S, T, σ, τ, and M and λ are varied in such a way that σ and τ stay at this value. If $\gamma = p + q$ and $\gamma \neq 0$, the interval $0 \le M < 2\pi\gamma$ covers the period of F as a function of M. The mean value required is then given by the expression

$$\bar{F} = \frac{1}{2\pi\gamma} \int_0^{2\pi\gamma} F \, d\, M = \frac{1}{2\pi\gamma} \int_0^{2\pi\gamma} F\, W \, d\, E \, , \quad \text{where E is the eccentric anomaly}$$

and $W = dM/dE = 1 - e \cos E$, as $E - e \sin E = M$. Only Δ^{-1} and ζ vary with E, so that it is sufficient to compute the mean values of $W\,\Delta^{-1}$ and $W\,\zeta$ with respect to E to get the mean value of F. A sufficiently fine division of at least 100 equidistant points covering the integration interval was used during the numerical computations to get the mean of the corresponding F values. With 100 points, the IBM-7094 electronic computer was able to deliver 350 values of the averaged function $\bar{F}$ in one minute.

The function $\bar{F}$ was computed as a function of the variables U, σ, τ, $\rho = \sqrt{|2S|}$, and i. The formulas used are the following:

$S = \pm \frac{1}{2} \rho^2$; S and $\gamma = p + q$ may have the same sign,

$$\sqrt{a} \cos \varphi = (qU - \gamma S)/(\gamma - p \cos i) ,$$

$$U - (S + T)\, p/q = \sqrt{a} = S + \sqrt{a} \cos \varphi = L ,$$

$$U - (S + T)(p + q)/q = \sqrt{a} \cos \varphi \cos i ,$$

$$e = \rho \sqrt{|L - \tfrac{1}{2} S|} \;/\; |L| ,$$

$$\omega = \tau - \sigma ,$$

$$M = E - e \sin E , \qquad 0 \le E < 2 \pi \gamma ,$$

$$A = L^2 (\cos E - e), \qquad B = L \sqrt{a} \cos \varphi \sin E ,$$

$$X = A \cos \omega - B \sin \omega , \; Y = A \sin \omega + B \cos \omega ,$$

$$\alpha = \ell_1 - \Omega = \tau + (M - \sigma)\, p/\gamma ,$$

$$\zeta = X \cos \alpha + Y \cos i \sin \alpha,$$

$$\Delta^{-1} = \left[(X - \cos \alpha)^2 + (Y \cos i - \sin \alpha)^2 + Y^2 \sin^2 i \right]^{-\frac{1}{2}},$$

$$W = 1 - e \cos E .$$

If F is replaced by $\bar{F}$ in the differential equations, then U becomes a constant. The same holds for the function $\bar{F}$. A constant $C = 1.5\, (\gamma n_1/p)^{2/3} - n_1 Uq/p$ was always subtracted from $\bar{F}$ to obtain small values of this function in the interesting cases.

For the planar case $i = 0$ some of the formulas become very simple. The value $\omega = 0$ was then used.

3. The long-period effects

A. The planar case

In the planar case $\bar{F}$ is a function of ρ and σ only, if the constant U has a fixed value. Curves $\bar{F}$ = const. can be drawn in a ρ,σ plane. For most commensurabilities, the expressions $x = \rho \cos \sigma$ and $y = \rho \sin \sigma$ were introduced to draw the curves $\bar{F}(x,y)$ = const. in an x,y coordinate system. This is the method used by Poincaré (1902a) and Andoyer (1903). Poincaré pointed out that when $m = 0$ the curves become circles around the origin. An extreme value or saddle point of $\bar{F}(x,y)$ corresponds to a periodic solution, because no long-period effects occur. There is always an extreme at the origin, if $m = 0$, while a second extreme is present on one of the circles for part of the U values. The last-mentioned corresponds to a commensurable mean motion. If the mass of the disturbing body is taken as m, a slight deformation of the surface $\bar{F}(x,y)$ will take place. One extreme will stay at or near the origin. If an extreme circle was present for $m = 0$, normally at least one extreme and one saddle point will stay near to it. The curves drawn by Poincaré represent a standard case for all commensurabilities in which $q = 1$. Thus a long-period oscillation is found for the semi-major axis and the eccentricity of the minor planet, both quantities being connected by the condition U = const. In many cases a libration is performed by σ.

Some simple qualities of $\bar{F}(x,y)$ will now be listed. As Poincaré pointed out, $\bar{F}$ will not change if y is replaced by -y. Therefore the plane $y = 0$ is a symmetry-plane for the surface $\bar{F}(x,y)$. A series development of $\bar{F}(\rho,\sigma)$, which contains only cosine terms with multiples of $q\sigma = qM - \gamma(\ell - \ell_1)$ as argument, is possible. Hence the expressions $\sigma = \pi n/q$; $n = 0, 1, 2, \ldots$; give a symmetry-plane for $\bar{F}(x,y)$. This indicates that when $q > 1$, one has $\frac{\partial F}{\partial x} = \frac{\partial F}{\partial y} = 0$ at $x = y = 0$. If more extremes or saddle points are present in the vicinity of the origin, they are fixed to the σ directions just mentioned. It can be shown with the aid of Newcomb operators that only a saddle point can appear at $\sigma = 180^\circ$, if the corresponding ρ is small and $q = 2,3,4,5$; $p > 0$ is optional. The series development of $\bar{F}(\rho,\sigma)$ is helpful in such a case for drawing the curves $\bar{F}(x,y)$ = const. The only curves found are those corresponding to the stable types among the figures published by Moser (1958) in a paper on the stability of the asteroids. If $q = 1$ or $q = 3$, for a special value of U a curve with a cusp is present. In the case $q = 1$ the cusp appears at the direction $\sigma = 180^\circ$, and if U is increased then, an extreme that is moving toward the origin and a saddle point moving outward begin to exist on the surface $\bar{F}(x,y)$, starting from the location of the cusp.

For the Hecuba commensurability ($p = q = 1$) Andoyer (1903) drew the curves $\bar{F}$ = const. in 6 cases, characterized by different values of U. The last two of his figures should not be used, because his series approximation was insufficient in those cases. His extremes with $y \neq 0$ corresponding to asymmetric periodic solutions disappear as soon as the accurate averaged function $\bar{F}$ is used. Even for large eccentricities, no asymmetric periodic orbits could be found in the Hecuba case or in any other corresponding to a mean motion larger than n_1.

Cases with a mean motion smaller than n_1 can be studied with $q < 0$, $\gamma > 0$, while $\gamma < 0$ allows the treatment of retrograde commensurable mean motions. Retrograde cases also can be treated with $\gamma > 0$ if the eccentricity angle has a value in the interval $90^\circ < \varphi < 180^\circ$, because a negative $\cos \varphi$ reverses the sense of the motion. This follows from the formulas for A and B. The common treatment of direct and of retrograde motion for a fixed commensurability ratio is possible with U and ρ values corresponding to the whole interval $0 \leq \varphi < 180^\circ$. The continuation of a class of periodic orbits, given by extremes for different U values, over a binary collision orbit ($\varphi = 90^\circ$) can be studied in this way.

If $\gamma < 0$ and $\cos \varphi < 0$, direct commensurable mean motions can be studied again. This allows one to handle the Trojan case with $p = 1$, $q = -2$. With $\gamma = -1$ one has $S \leq 0$ and therefore values of $\sqrt{a}$ in the vicinity of -1. Trojan orbits with a very small eccentricity cannot be included here, because $\varphi = 180^\circ$ must be excluded from this theory. But for this case the periodic orbits of Rabe (1961, 1962) are available to give the long-period effects.

The following commensurability cases were studied with the averaged function $\bar{F}$:

(p + q)/p	p	q	Figure numbers	Type
2/1	1	1	1 - 6	Hecuba
3/2	2	1	7 - 10	Hilda
4/3	3	1	11	Thule
3/1	1	2	12 - 16	Hestia
5/2	2	3	17	
7/3	3	4	-	
8/3	3	5	-	
9/4	4	5	-	
11/5	5	6	-	
-1/1	1	-2	22 - 24	Trojans
1/2	2	-1	18	
1/3	3	-2	19	
2/3	3	-1	20	
3/4	4	-1	21	Hyperion

The mass of Jupiter was used for all cases except the last one, where m = .00025 was taken as the mass ratio of the satellite Titan to its central body Saturn. This case is characteristic of the perturbations caused by Titan in the motion of the satellite Hyperion; compare the publication of Woltjer (1928).

For the more important cases a number of figures corresponding to different U values is given in Appendix 2. For q values larger than 3, almost all the curves $\bar{F}(x,y)$ = const. are nearly circular. Thus the oscillation of the elements a and e will be small and a libration of the angle σ is very improbable. All the figures published here were drawn with the aid of a data-plotting machine. Linear interpolation was made by a IBM-7094 program to get the points in the ρ,σ plane corresponding to a set of fixed values of $\bar{F}$, when the tabulation of $\bar{F}(\rho,\sigma)$ was given. ρ and σ were then taken as polar coordinates for the plotting. Therefore the canonical variables x and y are the rectangular coordinates. Only the curves for the Trojan case were drawn with ρ and σ as rectangular coordinates.

B. Nonplanar cases

If three-dimensional motion is treated, $\bar{\bar{F}}$ is a function of σ, τ, ρ, and i, while U is a constant. The problem is too complicated for a general discussion here, but some qualitative remarks are possible. If $m = 0$, the function $\bar{\bar{F}}$ depends only on $S + T$. For appropriate values of U it has a minimum at a special value of $S + T$, which corresponds to a commensurable mean motion. If $m \neq 0$, the combination $S + T = \sqrt{a}\,(1 - \cos\varphi \cos i)$ will be restricted, because $\bar{\bar{F}}$ will be constant and depends mainly on $S + T$. Since the constant U gives a second condition for $\sqrt{a}$ and $\cos\varphi \cos i$, these quantities will be restricted to finite intervals. As in the planar case, it does not imply instability, according to this theory, if the existing minor-planet orbits are considered.

Nonplanar periodic orbits are indicated, if the four independent variables can be chosen in such a way that the partial derivatives of $\bar{\bar{F}}$ become equal to zero. Some simple qualities of $\bar{\bar{F}}$ are very useful for this. $\bar{\bar{F}}$ does not change if σ and τ change sign at the same moment. If τ is augmented by π or σ by 2π, $\bar{\bar{F}}$ remains unchanged. If q is an even number, σ also can be augmented by π without a change in $\bar{\bar{F}}$. This can be proved with the aid of the series development of the disturbing function. If $\bar{\bar{F}}$ is now treated as a power series in x, y, ξ, η, only special combinations of the variables have nonzero coefficients. If q is even, it can be shown that $x = y = 0$ is a particular solution of the differential equations. The long-range effects in ξ and η can be studied then by drawing the curves $\bar{\bar{F}}(\xi,\eta) = \text{const}$. Periodic solutions will appear on the ξ and η axes; one always is present at the origin, where $i = 0$. This was found when the case $p = 1$, $q = 2$ with $U = 0.696$ was treated as an example, while no periodic solutions could be detected for $p = q = 1$, $U = 0.8$ except the planar ones. But in the case $p = 2$, $q = 3$, $U = 0.74$, nonplanar periodic solutions exist with $y = 0$ and an x value close to zero.

C. The effects as a function of time

So far, only effects have been mentioned that could be evaluated with the aid of the integral $\bar{\bar{F}} = \text{const.}$ of the equations of motion. To obtain the dependence on time, numerical integration of the differential equations is necessary. The partial derivatives of $\bar{\bar{F}}$ are needed for this. They also can be obtained by the averaging process described for $\bar{\bar{F}}$, but a series development was used here for the computation of the derivatives of $\bar{\bar{F}}$ in the planar case, if the values of ρ were sufficiently small. An IBM-7094 program was prepared to do the integration of x, y, and ℓ as functions of time, if the series development is known. Laplace coefficients (Izsak and Benima, 1963) and special Newcomb operators for canonical elements, prepared by Izsak and others, are available to get the development of $\bar{\bar{F}}$. Another IBM-7094 program doing the accurate numerical integration of any minor-planet case of the three-body problem in rectangular coordinates was used to check the theory described here and to indicate its limitations. It gave the most reliable way to get the time dependence of the long-period effects in special cases.

If a saddle point is present for the surface $\bar{F}$, the corresponding periodic solution will be unstable. Asymptotic orbits will start from such a solution. The time required to approach or to depart from the unstable periodic solution is infinite. An extreme of the function $\bar{F}$ gives a stable periodic solution. The forms of motion in the vicinity of it can be obtained with a variation theory. The necessary second-order derivatives of $\bar{F}$ were computed by numerical differentiation. The Hecuba and Hilda commensurabilities were treated in this way to get the periods of the long-range effects in the planar case. Orbits that differ more widely from a periodic solution will have periods of the same order of magnitude, unless they are close to an asymptotic orbit. A list of stable periodic solutions with periods of the variational orbits is given here.

HECUBA $\quad p = 1 \quad q = 1 \quad \sigma = 0$

U	ρ	$\sqrt{a}$	e	cos φ	Period (Jupiter = 1)
.793	.058	.791	.065	1.00	37
.80	.121	.793	.136	.99	39
.82	.231	.793	.257	.97	37
.87	.391	.794	.43	.90	36
.96	.577	.794	.61	.79	34
1.1	.783	.794	.79	.61	30
1.3	1.006	.794	.93	.36	24
1.5	1.188	.794	.99	.11	20
1.7	1.346	.794	.99	-.14	17
1.9	1.487	.794	.92	-.39	17
2.1	1.616	.794	.76	-.64	21
2.3	1.736	.793	.44	-.90	57

HILDA p = 2 q = 1 σ = 0

U	ρ	√a	e	cos φ	Period (Jupiter = 1)
.882	.098	.872	.105	.99	23
.887	.120	.873	.128	.99	23
.90	.164	.873	.175	.98	23
.92	.216	.873	.230	.97	22
.95	.277	.873	.293	.96	22
1.0	.356	.874	.374	.93	21
1.1	.476	.874	.492	.87	19
1.2	.571	.874	.582	.81	16
1.4	.725	.874	.715	.70	10
1.6	.852	.874	.811	.58	6

The mass of Jupiter was used here and the periods given are multiples of the orbital period of Jupiter. In this unit the synodic periods of Hecuba and Hilda are close to 1 and 2. The periodic solutions found here approximately as minima of $\bar{\bar{F}}$ (ρ,σ) really exist, as could be shown by accurate numerical integration for some of the cases. Schwarzschild (1898) treated these periodic solutions, and Poincaré (1902b) mentioned their connection with the set of noncommensurable nearly circular periodic orbits. This set is interrupted at commensurabilities with q = 1, as Poincaré pointed out. His predictions can be proved with the theory given here. The list given here indicates that in the Hecuba case, a connection exists between very eccentric and retrograde periodic orbits.

The periods of the long-range effects of some accurately computed minor-planet orbits proved to be about equal to or slightly smaller than the corresponding value in the lists. For the case p = q = 1, U = 0.8 (see figure 3), the variables x and y were obtained as functions of time, as described. By taking start-values from the relations $x = x_0$, $y = 0$, six curves were obtained which surround the point corresponding to the periodic solution and are enclosed by the curve corresponding to an asymptotic orbit. The periods are given here in the same unit as before:

x_0	Period (Jupiter = 1)
+0.01	49.9
+0.03	41.8
+0.05	39.0
+0.07	38.2
+0.09	38.5
+0.11	39.0

To give a rough estimate, the period of the long-range effects is equal to about 40 periods of Jupiter for Hecuba-type orbits and to about 20 for Hilda-like cases. For the Trojan planets the corresponding value is known to be 13 (see Rabe, 1962). This corresponds to about 500, 250, and 150 years in the three cases respectively.

4. Application to real cases

A. The gaps

A statistical explanation for the gaps in the distribution of the mean motions of the asteroids was given by Brouwer (1963). His conclusions are based on a theory that is adapted to the single commensurability cases, as was done here. This theory indicates a gap for all commensurability cases. While large gaps are present for the commensurability ratios 2/1 and 3/1, and less important ones for ratios between these with larger q values, an accumulation of minor planets is found for the ratio 3/2. This is the very well-defined Hilda group. To explain the difference between this case and the others with gaps, Brouwer pointed out that there are many more less-important commensurabilities crowded around the 3/2 case than around the 2/1 case, if equal intervals in the mean motion are considered and commensurabilities with q up to 9 are marked in them. This is not an explanation if the oscillation in the mean motion caused by the commensurability is smaller in the 3/2 case than in the 2/1 case. But otherwise the theory can fail for the Hilda case, as the nearby commensurabilities can temporarily become important and must be included in any theory of the long-range effects.

The oscillations in the mean motions for the Hecuba and Hilda case can be found from drawings such as those in figures 1-10, if the formula $\sqrt{a} = U - .5\,\rho^2\, p/q$ is used. Maximum values of the oscillation found for some U values are listed here. The difference Δn of the extremes of the mean motion is given in seconds of arc per day, together with the corresponding extremes of ρ. The mean motion of Jupiter is about 299" in the same unit.

HECUBA $p = 1$ $q = 1$

U	ρ_1	ρ_2	Δn
.795	.02	.13	19"
.80	.00	.18	36"
.809	.10	.23	51"
.82	.16	.28	61"
.87	.33	.44	93"

HILDA $p = 2$ $q = 1$

U	ρ_1	ρ_2	Δn
.88	.01	.13	27"
.885	.02	.16	39"
.895	.07	.20	52"
.91	.12	.24	66"
.93	.17	.29	83"

The oscillation is found to increase with the mean eccentricity, characterized by the mean of the two values of ρ given. As the Hilda case offers the larger oscillations, Brouwer's explanation is supported. It might be added that Hilda-type planets are more sensitive to additional effects, as moderate eccentricities can cause close approaches to Jupiter.

Figures 1-6 in Appendix 2 give an impression of the oscillation in ρ for the Hecuba case. Figures 3 and 4 correspond to figures 9 and 12 respectively in the paper by Andoyer (1903), but in the work reported in the present paper no asymmetric periodic solutions were found for this case. The periodic solutions with $\sigma = 0$ have been listed previously. The unstable periodic solutions with $\sigma = 180^\circ$ are connected with nearly circular ones, as is shown in figures 2, 3, and 4. In the introduction an example was given for a minor planet, whose mean motion passes through the Hecuba gap. An interesting question is: how many of the numbered asteroids will do so during a 500-year period?

Figures 12-16 are given for planets close to the Hestia gap. A periodic orbit always is given at the origin. It is circular in this approximation. While eccentric periodic orbits begin to develop from it, it is found to be unstable for a short interval in U (see also v. Zeipel, 1915). Figure 17 is related to the 5/2 commensurability ratio, which corresponds to the largest gap between the before-mentioned Hecuba and Hestia gaps. According to the symmetries of this case, only two types of periodic solutions begin to develop from the one that is almost circular. This holds for every $q > 1$ examined here.

B. The groups

The curves shown in figures 7-10 for the Hilda case are analogous to those for the Hecuba commensurability. For the asteroids of the Hilda group the theory developed here will hold only if the oscillations in the semi-major axes remain small. This will be true for cases with a libration in σ around zero with small amplitude. It is interesting to look at the planets with large eccentricities, where σ is well defined. The list given here was computed with the aid of tables of just these objects published by Tshebotarev and Boshkova (1953). Two objects were added, one of which is Comet Oterma (Marsden, 1961), that temporarily belonged to the Hilda group. The elements of the comet for 1950 were used. The mean motions are listed in seconds of arc per day.

Object		σ	φ	i	Mean motion
153	HILDA	-38°	9°	8°	450"
190	ISMENE	4	10	6	453
361	BONONIA	40	12	13	457
499	VENUSIA	-34	13	2	452
748	SIMEISA	65	10	2	453
958	ASPLINDA	11	11	6	455
1038	TUCKIA	26	14	9	456
1162	LARISSA	22	6	2	450
1180	RITA	-58	10	7	449

Object		σ	φ	i	Mean motion
1202	MARINA	-5°	12°	3°	458"
1212	FRANCETTE	35	10	8	448
1268	LIBYA	32	6	4	455
1345	POTOMAC	-7	10	11	450
1439	VOGTIA	-29	7	4	451
1512	1939 FE	14	10	7	451
1529	1938 BC	-53	11	9	444
1578	KIRKWOOD	72	13	1	451
--	OTERMA	-172	8	4	448

The list shows σ values surrounding 0° for all objects except the comet. As it is improbable that this is due to the epoch for so many cases, one can expect libration for all asteroids of the Hilda group, and the stability of the group is indicated by the theory. Small values of $|\sigma|$ prevent close approaches to Jupiter, while values around 180° allow them. This difference between the Hilda-asteroids and Comet Oterma was pointed out by Marsden (1962). Indeed, a close approach of the comet to Jupiter in 1963 completely changed the orbit, so that the theory given here is disproved for that case. It seems to be important to prove it for at least one of the asteroid orbits, as Tshebotarev (1953) calculated that a complete disintegration of the Hilda group would occur within the next 1000 years. He used a periodic orbit and variational equations. In the present paper, accurate numerical integration on the IBM-7094 computer was used to find the variations of the semi-major axes of two test orbits that are similar to the orbit of the planet Hilda. 23000 integration steps were done in both cases to cover a 1000-year interval. In the first case the eccentricity of Jupiter and the inclination were neglected, while in the second case these elements were taken into account. After the short-period effects are smoothed out, a quite regular, sinusoidal oscillation remains in the semi-major axes taken as function of the time. In the first case a oscillates between 0.756 and 0.767, while in the second the amplitude is only one-half as large. The periods of the oscillations confirm the values given by the theory. This means that Tshebotarev's method cannot be applied to such a long time interval. Stability is indicated for the whole Hilda group of asteroids. A numerical integration to show the libration of σ for Hilda already has been done by Hirayama and Akiyama (1937). Akiyama (1962) published a continuation of that integration. As he takes a noncircular orbit for Jupiter, the libration of σ is influenced by additional effects.

The minor planet Thule is the only object known as an example of the $4/3$ commensurability. As its eccentricity is very small, the theory indicates a small oscillation in the semi-major axis and thus stability; compare figure 11. The same will hold for the Hilda-group members with small eccentricity. No other minor planets have mean motions in this range except the ones belonging to a group.

Groups belonging to the $1/1$ commensurability are represented by the Trojan planets and Jupiter's satellites. The latter will not be treated here. The last three figures given in Appendix 2 are related to the Trojan case. Figures 22, 23, and 24 are given for $\varphi = 170^\circ$, 160°, and 150° respectively, as $U = -\cos^2 \frac{\varphi}{2}$ with $p = 1$, $q = -2$, and $L = -1$. Thus the last figure corresponds to an eccentricity varying around 0.5 and a semi-major axis close to 1, while $\cos \varphi$, $\sqrt{a}$, and the mean motion are negative. 2σ is equivalent to the mean angular difference between Jupiter and the Trojan as seen from the sun. Therefore a periodic orbit can be expected for $\sigma \approx 30^\circ$; compare the paper by Willard (1913). In the figures, σ was used as abscissa, ρ as ordinate. A 90° interval is given in σ, starting with $\sigma = 13^\circ$, 23°, and 33° respectively, in figures 22, 23, and 24. As can be seen from the figures, the expected periodic solutions are stable. The corresponding σ-values become larger with increasing eccentricity. An unstable periodic solution is present for $\sigma = 90^\circ$. The curves are symmetric with respect to $\sigma = 90^\circ$.

Libration is possible around the stable periodic solutions, in analogy to the well-known forms of motion with small eccentricity. Orbits with σ passing the value 90° are indicated in figures 22 and 23. They close symmetrically around a second stable periodic solution in analogy to the horseshoe-shaped orbits described by Rabe (1961). Asymptotic forms of motion start or end at the unstable periodic solution, but it is by no means proved that an orbit starting there also will end there asymptotically, as this theory is only an approximation. Some Trojan orbits can even change from a libration of large amplitude around one of the stable periodic solutions to the other form of motion with σ passing 90°. Thüring (1959) found this for disturbing masses larger than Jupiter, using accurate numerical integration. Rabe and Schubart confirmed this for the mass value of Jupiter by an unpublished example integrated with the computer of the University in Heidelberg, Germany. Such a change from libration to nonlibration, effected by crossing the curve that corresponds to an asymptotic solution, may also be possible for other commensurability cases.

C. The outer commensurability cases

There is only one minor planet, Hidalgo, with a semi-major axis considerably larger than 1. But it is not dominated by a single commensurability ratio and has a very eccentric orbit. An example of a commensurability with the disturbing mass on the inner orbit and the disturbed mass on the outer is given by two Saturn satellites, as mentioned before. Figure 21 was drawn by use of the commensurability ratio 3/4 and the mass $m = 1/4000$ corresponding to that case. Figures 18, 19, and 20 refer to a small object outside the orbit of Jupiter and represent the three most important outer-commensurability ratios.

Figures 20 and 21 both show the possibility of a libration of σ about 180°, which is indeed realized in the case of the Saturn satellites. Then the oscillations in the mean motion remain small, and close approaches to the disturbing body are avoided, as was found for the Hilda group.

Figures 18 and 19 contain asymmetric periodic orbits. For the commensurability ratio 1/2, such orbits have already been found and proved by Message (1958, 1959). For the ratio 1/3 no direct proof by numerical integration for an asymmetric solution has yet been done. As the Trojan case with the ratio 1/1 in the stable periodic solutions also contains asymmetric periodic orbits, it is indicated that ratios with the numerator 1 favor the occurrence of this type of solution. Cases with $p + q = \pm 1$ differ from all others by the fact that only then the mean value of ζ used for the computation of $\bar{F}$ is different from zero. Thus the influence of ζ seems to produce the asymmetric periodic orbits.

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Appendix 1: Canonical Variables

The following equations given without a proof suggest new systems of canonical variables. These are useful, if $\cos\varphi$ is small or equal zero. A transformation to separate short-period and critical arguments also can be done here.

$$\sqrt{a}\, dM + \sqrt{a}\cos\varphi\, d\omega + \sqrt{a}\cos\varphi\cos i\, d\Omega$$

$$= \sqrt{a}\, dM + \sqrt{a}\cos\varphi\cos\alpha_1\, d\beta + \sqrt{a}\cos\varphi\cos i\, d\psi$$

$$= \sqrt{a}\, dM + \sqrt{a}\cos\varphi\, P_z Q_x\, d\,(P_x/P_z) + \sqrt{a}\cos\varphi\, P_z Q_y\, d(P_y/P_z)$$

$$= \sqrt{a}\, dM + \sqrt{a}\cos\varphi\,(Q_x + R_y)\, d\,[P_x/(1 + P_z)]$$

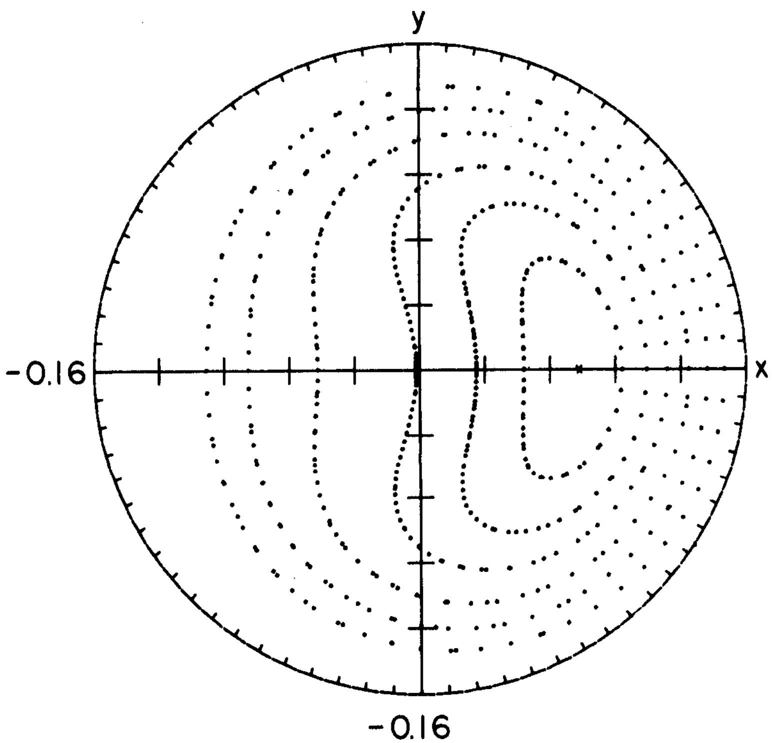
$$+ \sqrt{a}\cos\varphi\,(Q_y - R_x)\, d\,[P_y/(1 + P_z)]\ .$$

$$\sqrt{a}\, dM + \sqrt{a}\cos\varphi\, d\omega + \sqrt{a}\cos\varphi\cos i\, d(\Omega - \ell_1)$$

$$= \sqrt{a}\, dM + \sqrt{a}\cos\varphi\cos\alpha_1\, d\beta + \sqrt{a}\cos\varphi\cos i\, d(\psi - \ell_1)$$

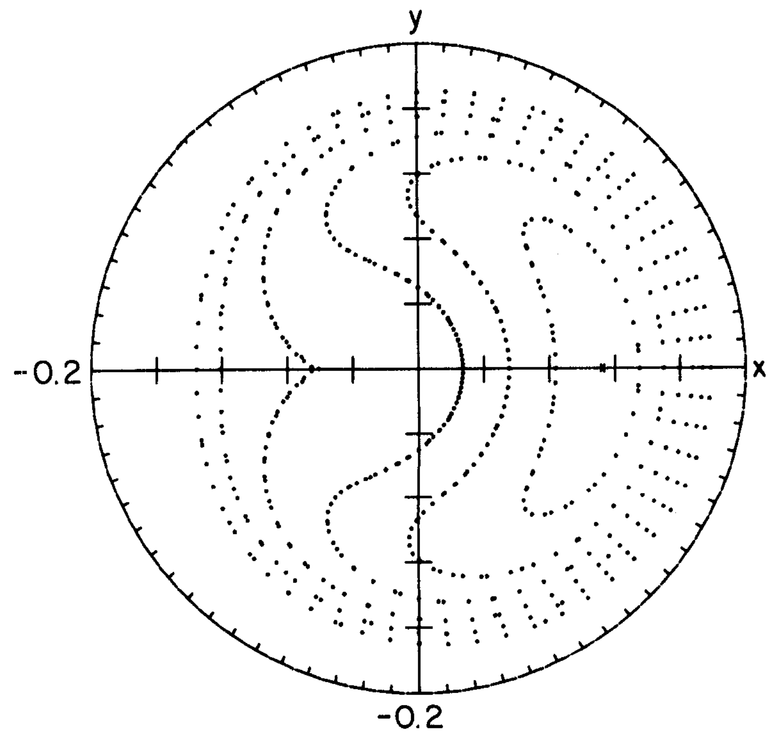
$$= U\, d(qM/\gamma) + \sqrt{a}\cos\varphi\cos\alpha_1\, d\beta + \sqrt{a}\cos\varphi\cos i\, d[\psi - \ell_1 + pM/\gamma]\ .$$

$\omega = \tau - \sigma$ is the distance of the perihelion from the ascending node. $\vec{P} = (P_x, P_y, P_z) = (\cos\beta\cos\psi\ ,\ \cos\beta\sin\psi\ ,\ \sin\beta)$ is the unit vector of the direction to the perihelion if Jupiter moves in the x,y plane. $\vec{Q}$ and $\vec{R}$ also are unit vectors with $(\vec{P}\,\vec{Q}) = 0$, $\vec{R} = [\vec{P} \times \vec{Q}]$ and $\vec{Q}$ giving the direction to the object, if the true anomaly is 90°. α_1 is defined by $\cos\alpha_1 = \sin i \cos(\psi - \Omega)$.



$p=1,\ q=1,\ U=0.795$

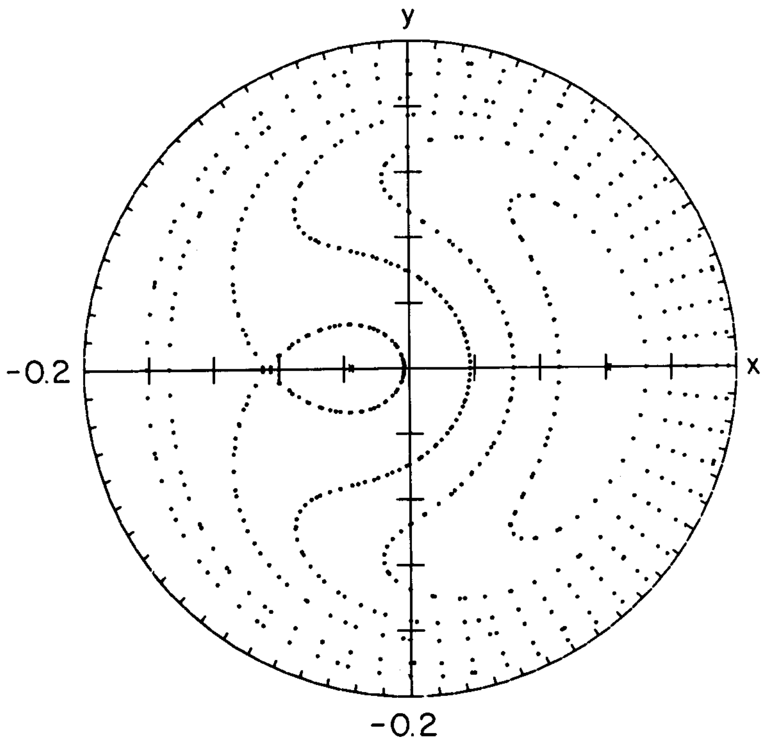
Figure 1



$p=1,\ q=1,\ U=0.79886$

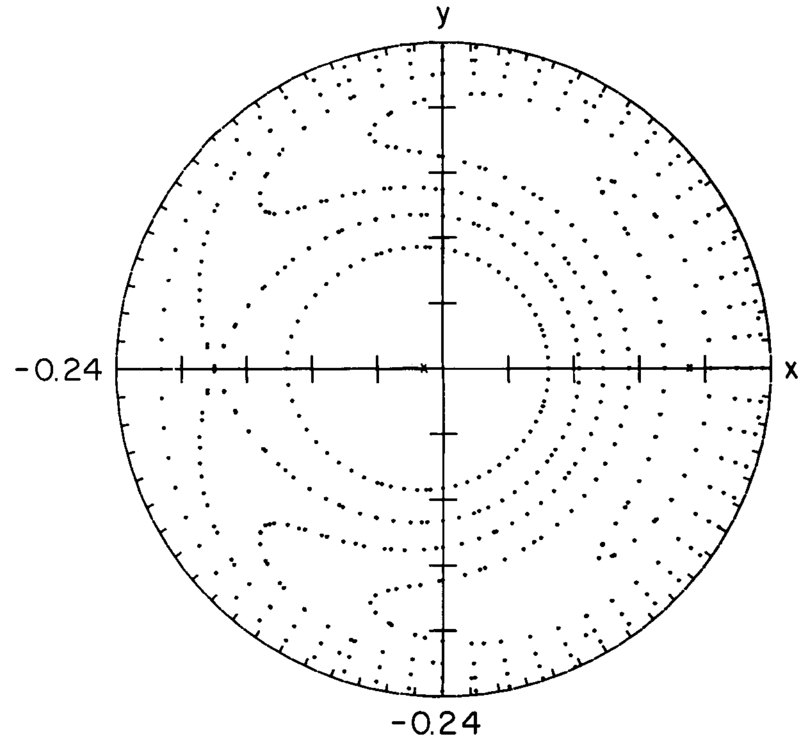
Figure 2

(Explanations of the figures are found in the text, in the chart on page 9.)



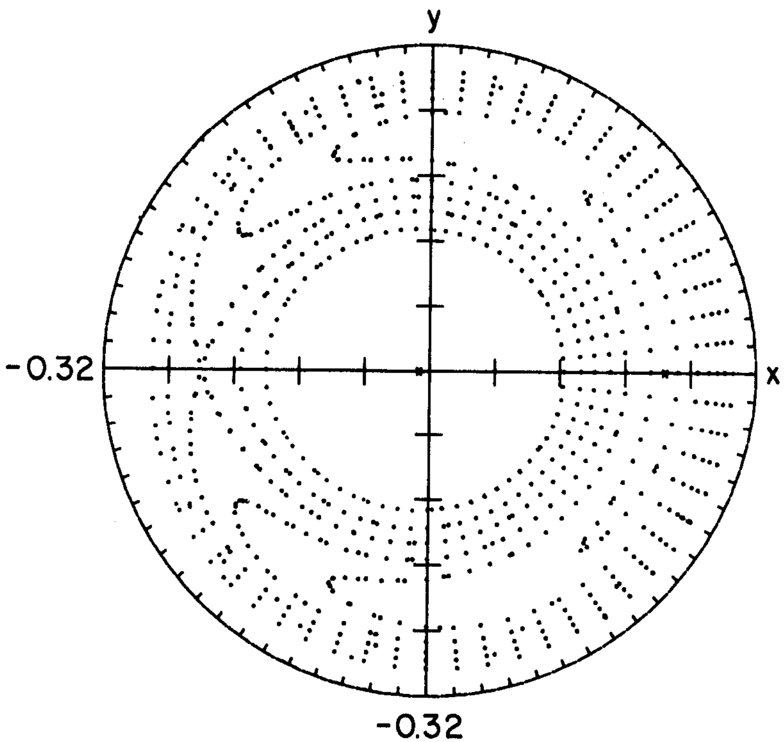
p=1, q=1, U=0.8

Figure 3



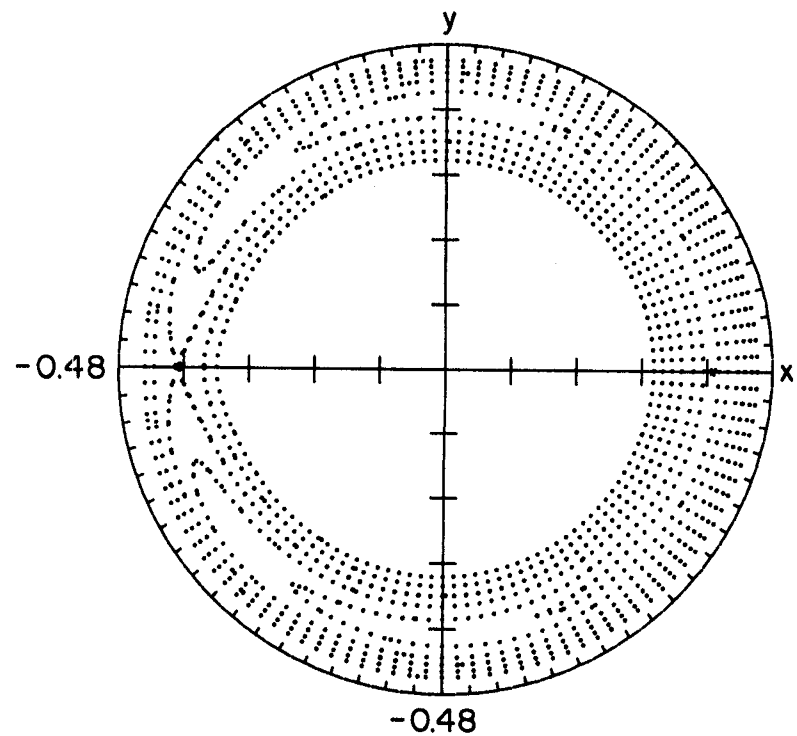
p=1, q=1, U=0.8093207

Figure 4



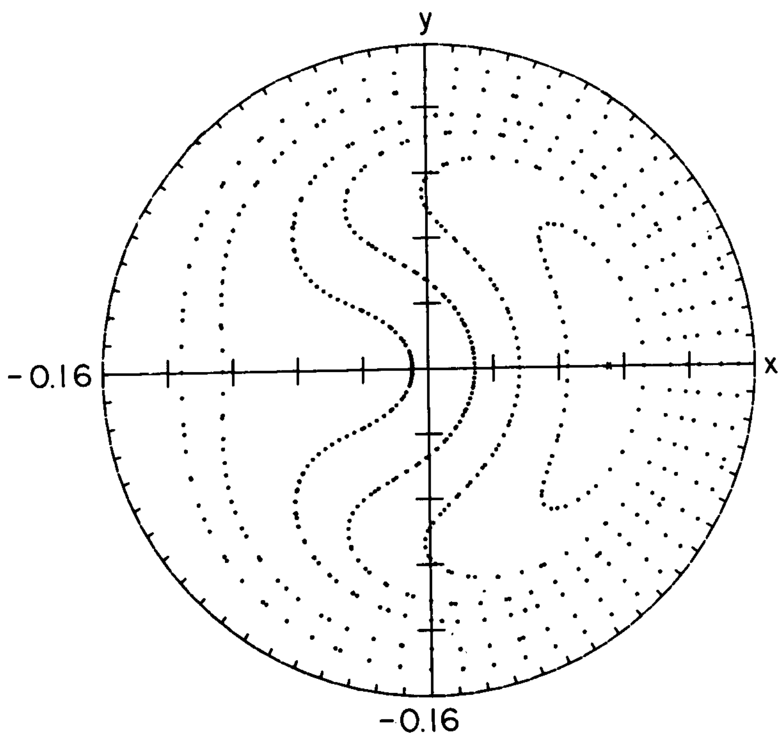
p=1, q=1, U=0.82

Figure 5



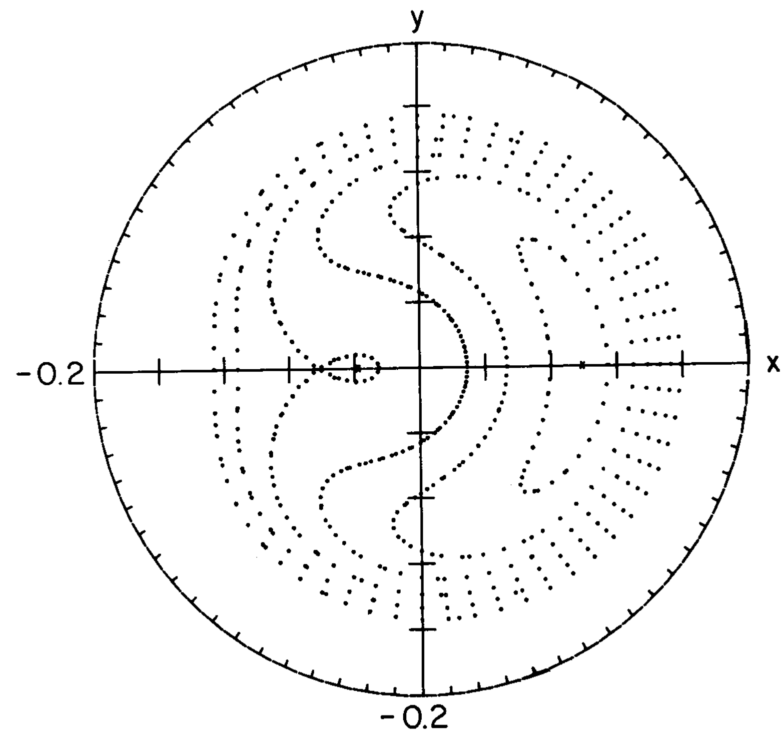
p=1, q=1, U=0.87

Figure 6



p=2, q=1, U=0.88

Figure 7



p=2, q=1, U=0.882

Figure 8

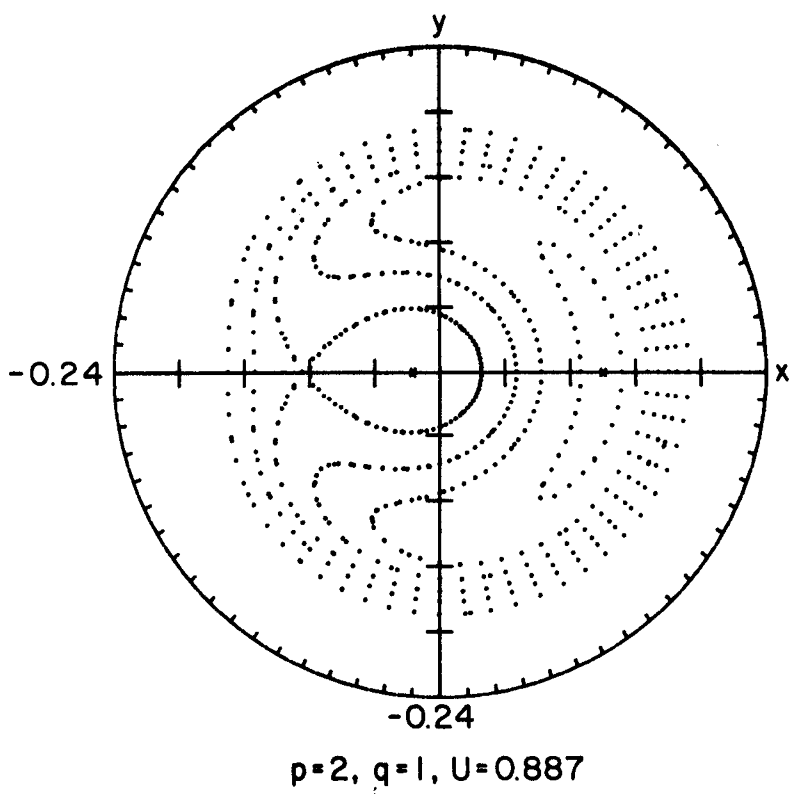


Figure 9

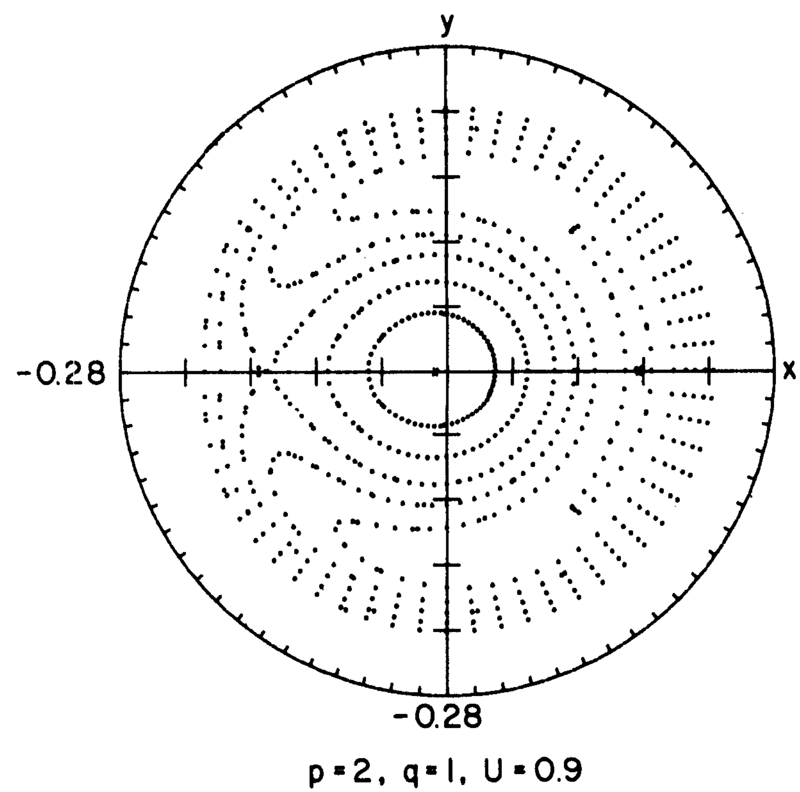
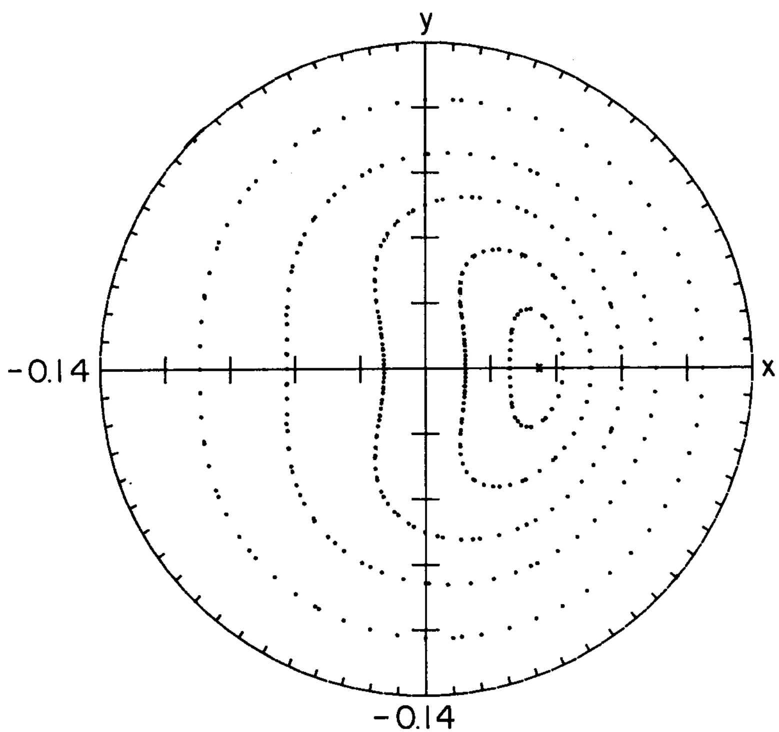
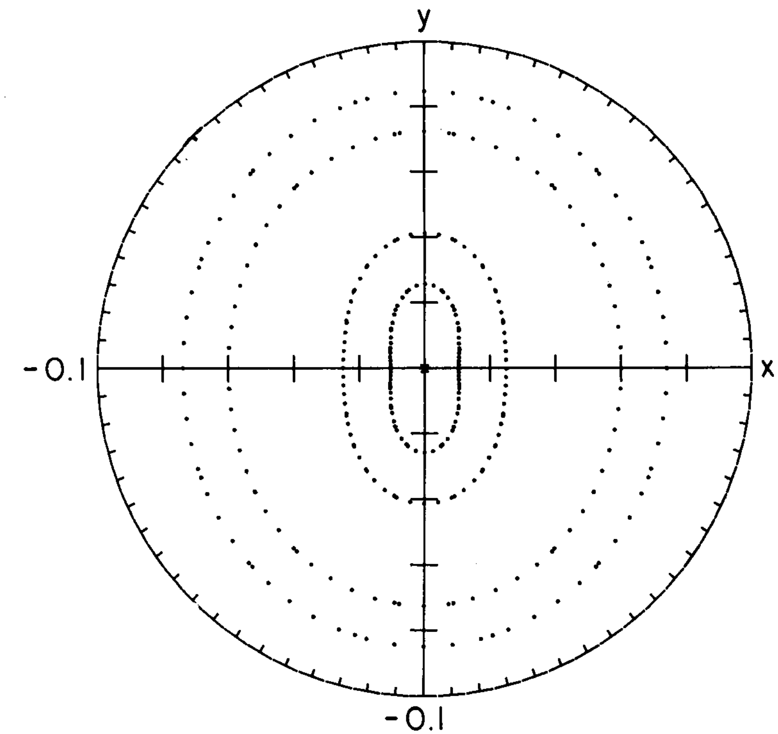


Figure 10



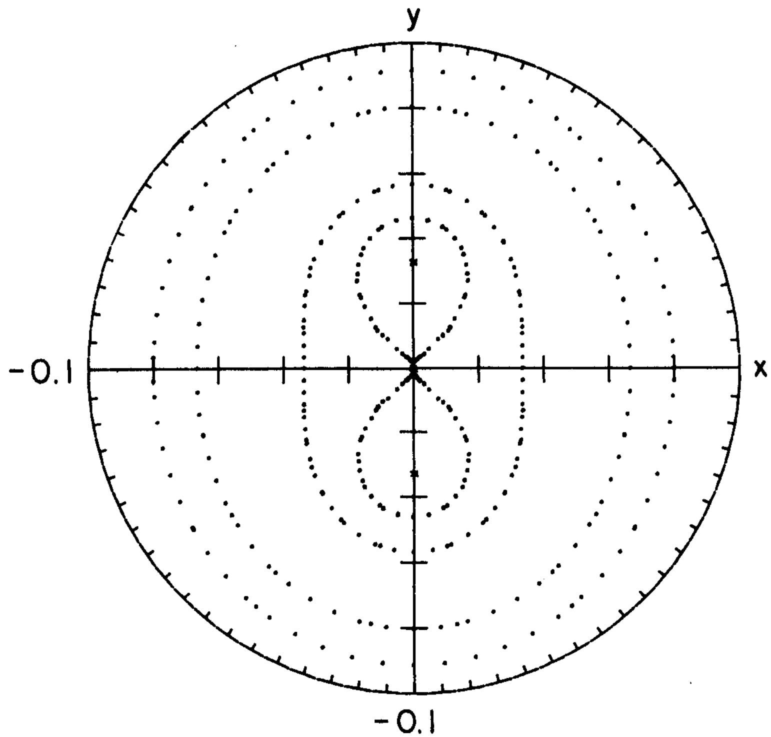
p=3, q=1, U=0.909

Figure 11



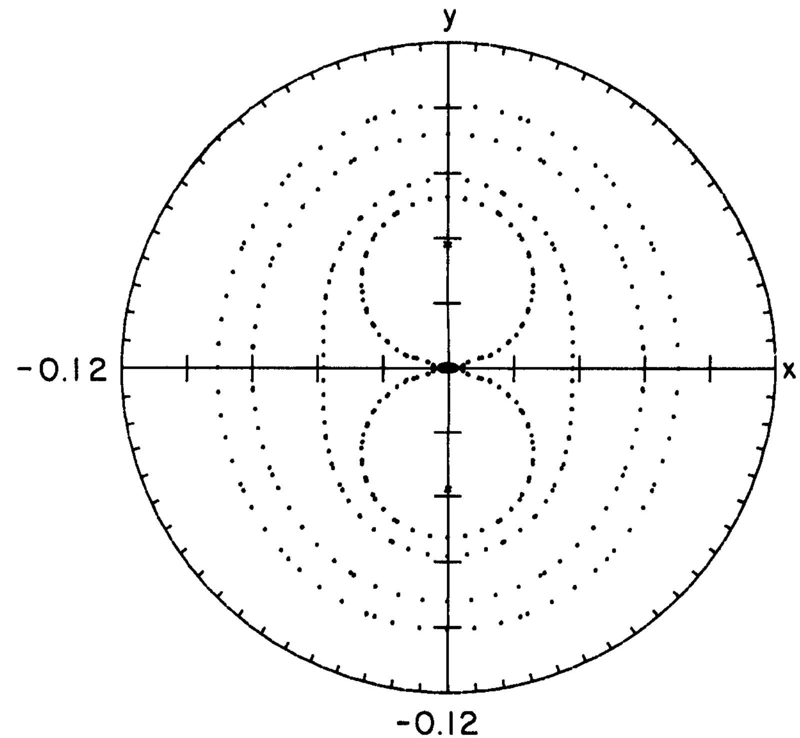
p=1, q=2, U=0.693025

Figure 12



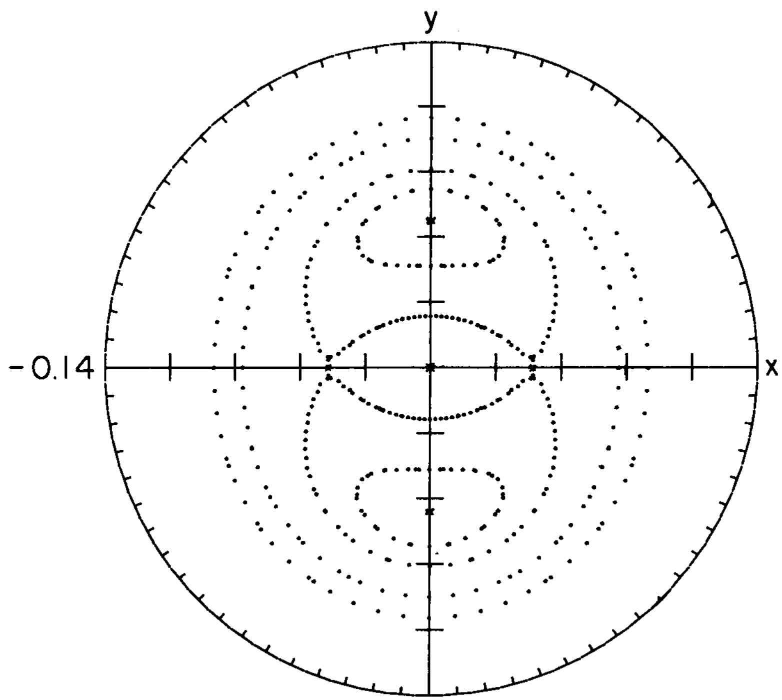
p=1, q=2, U=0.6933

Figure 13



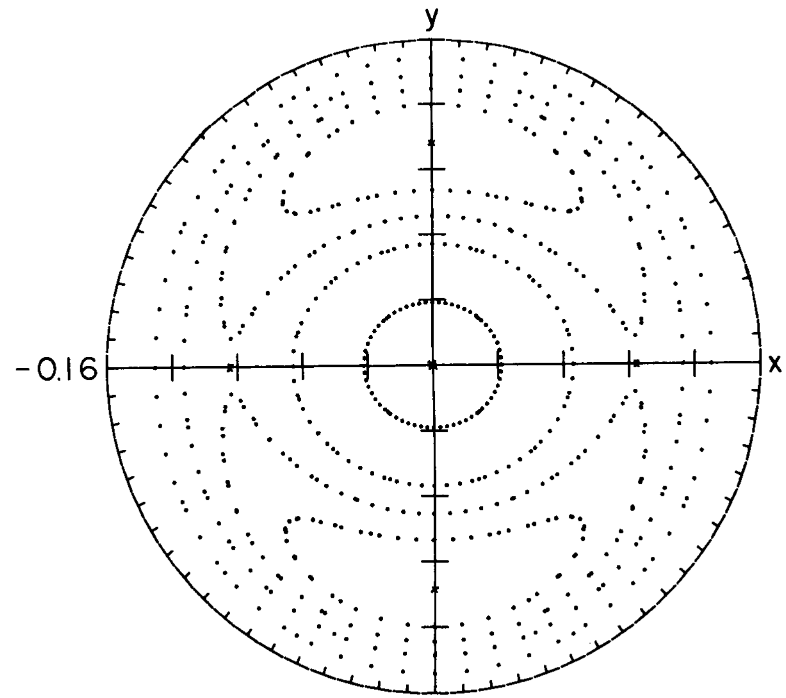
p=1, q=2, U=0.693532

Figure 14



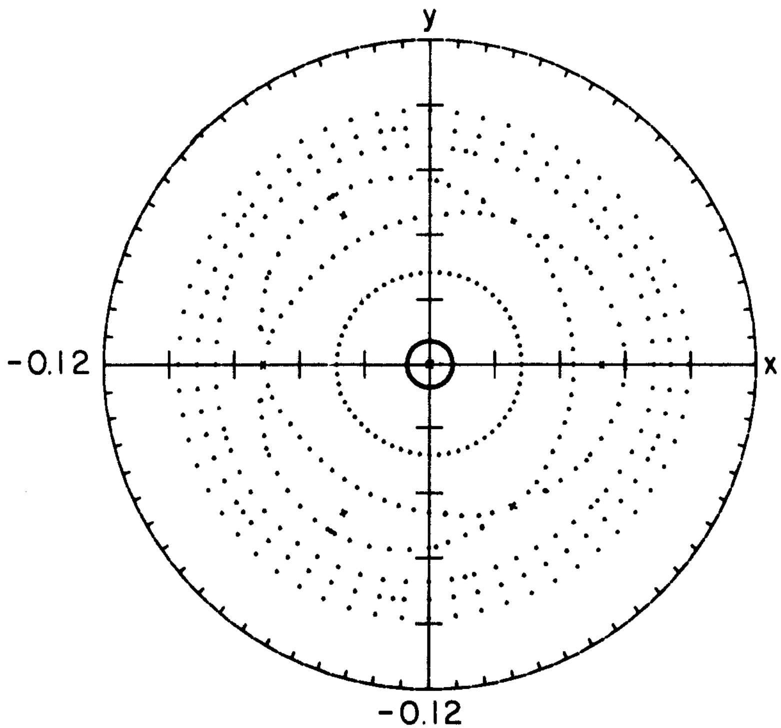
p=1, q=2, U=0.694

Figure 15



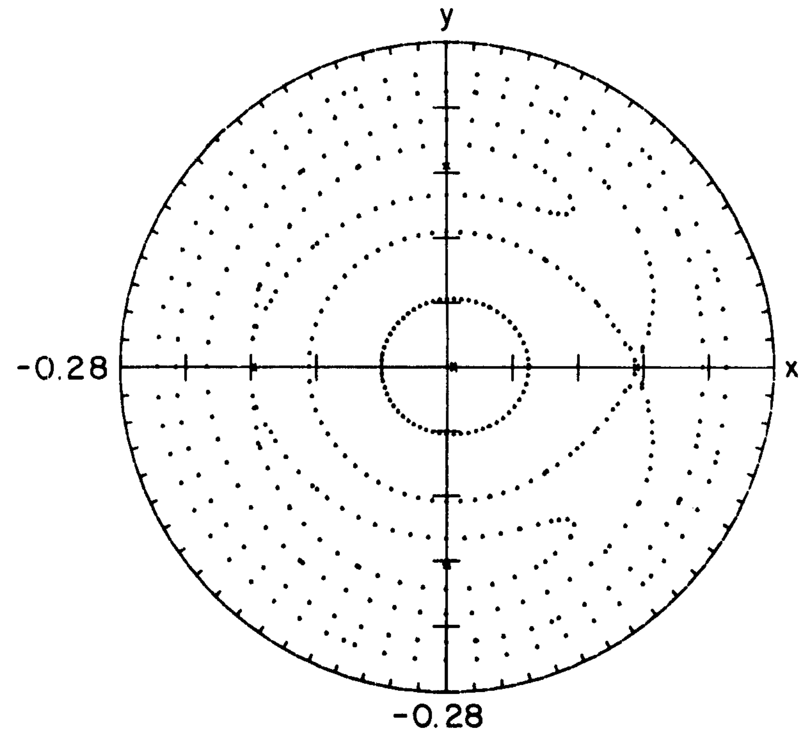
p=1, q=2, U=0.696

Figure 16



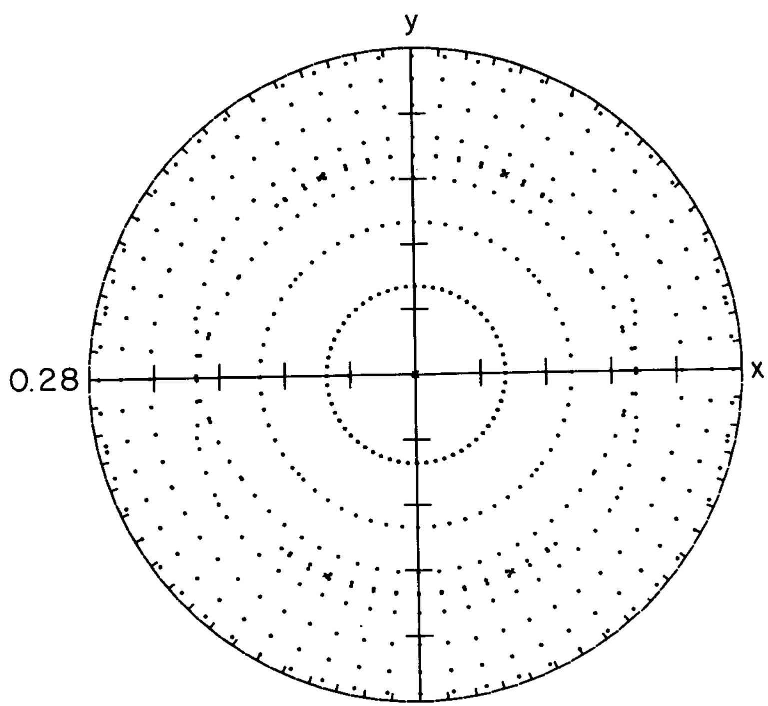
$p = 2,\ q = 3,\ U = 0.738$

Figure 17



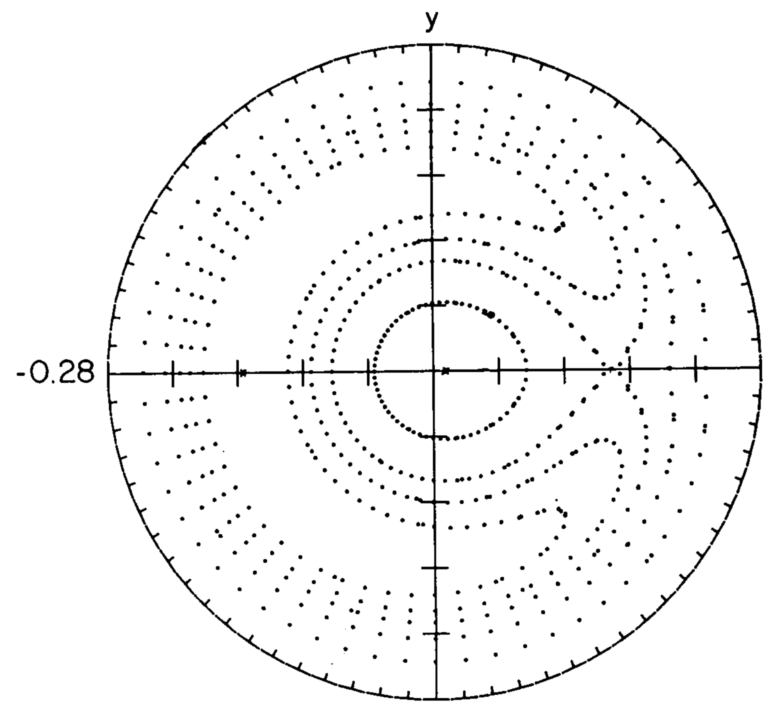
$p = 2,\ q = -1,\ U = 1.232$

Figure 18



p = 3, q = -2, U = 1.418

Figure 19



p=3, q=-1, U=1.106

Figure 20

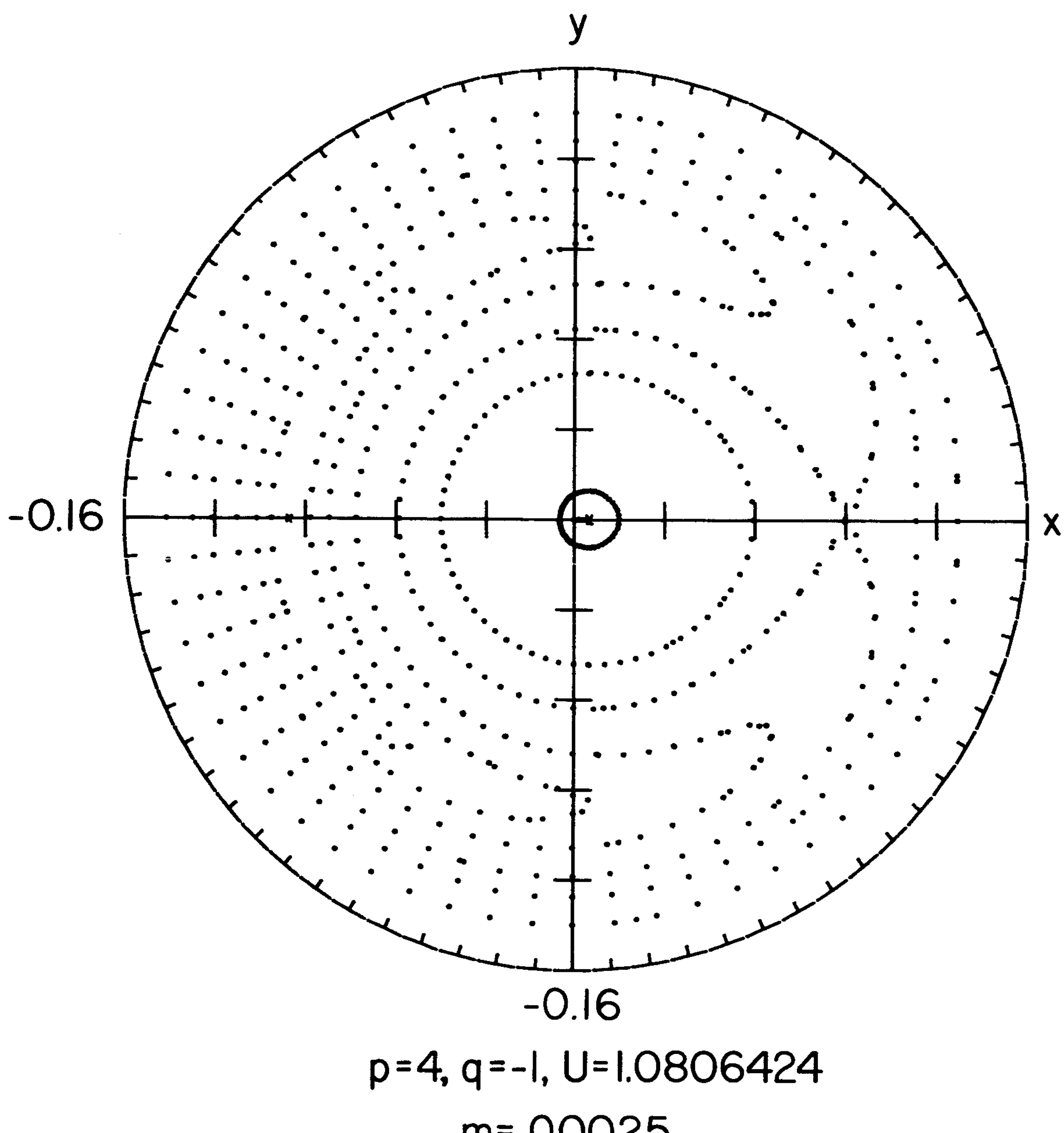
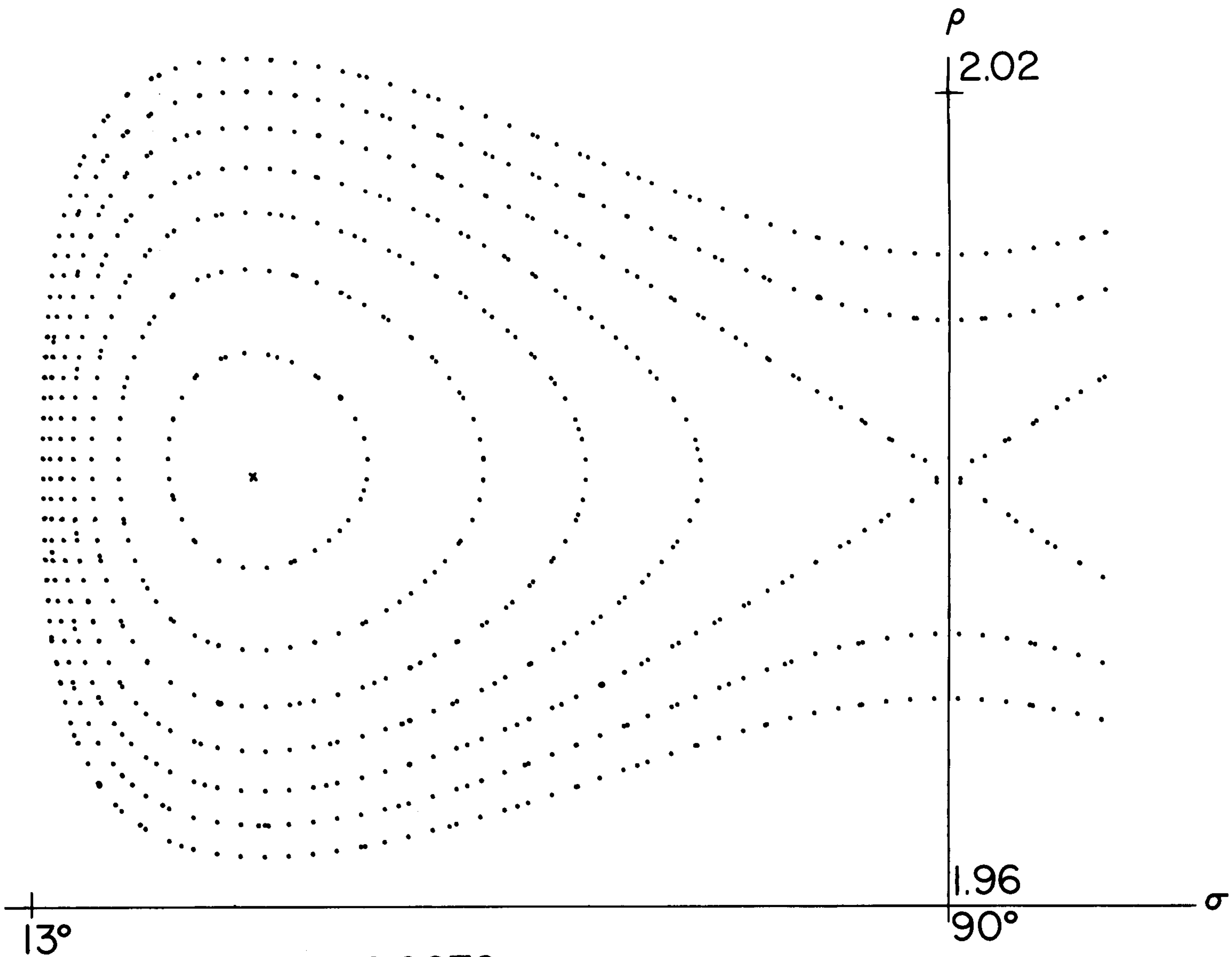


Figure 21



p=1, q=-2, U=-0.0076

Figure 22

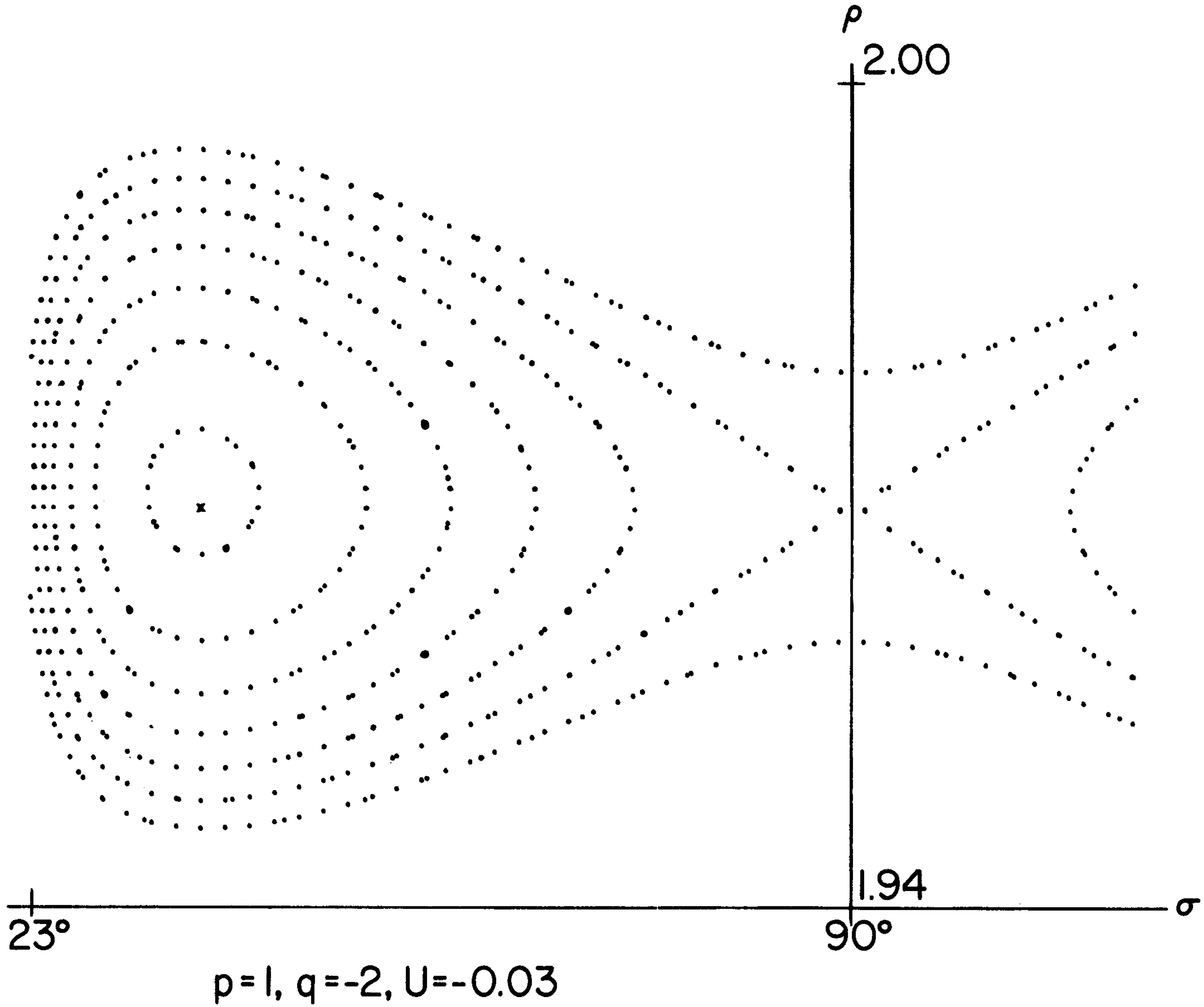
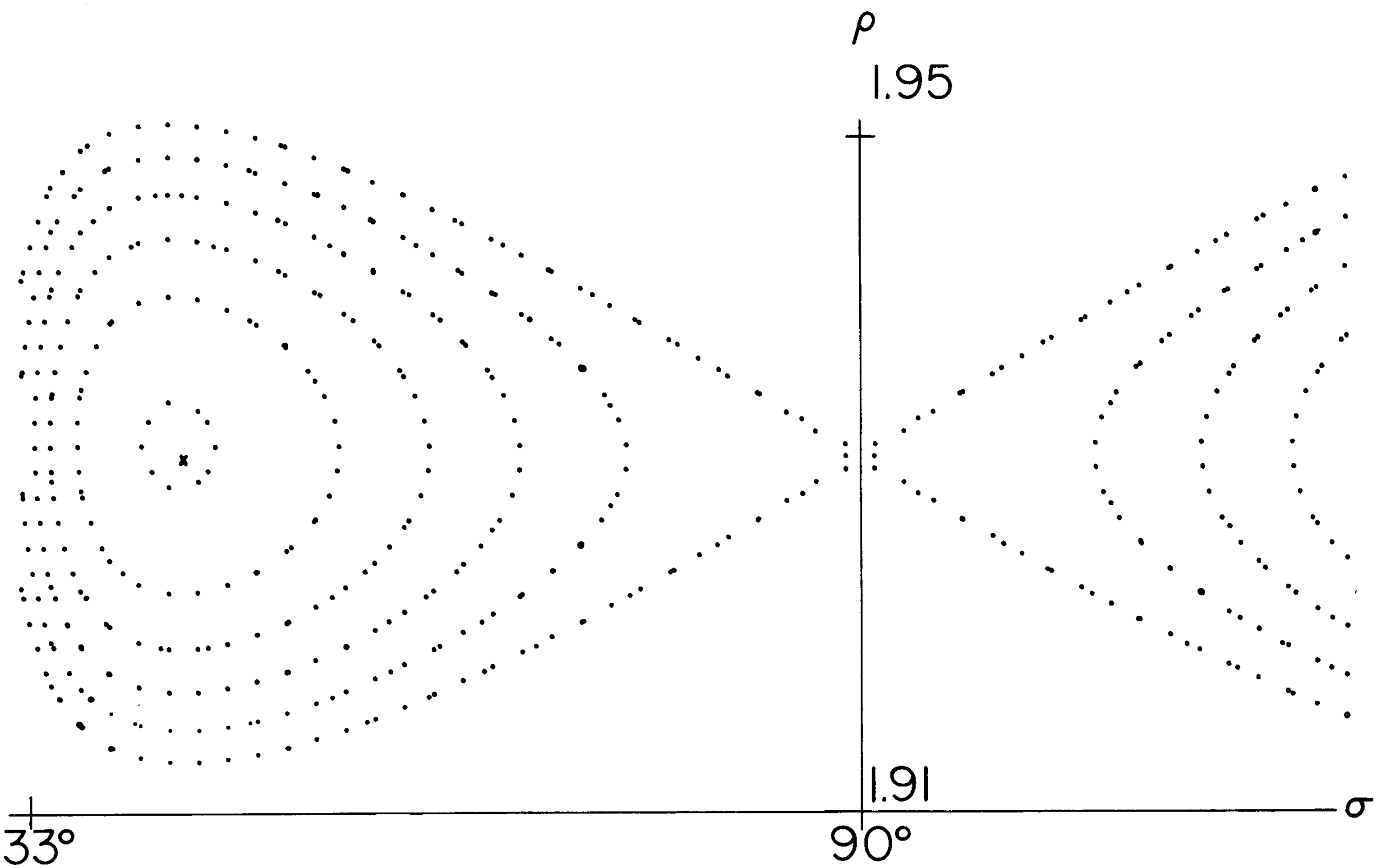


Figure 23



$p=1,\ q=-2,\ U=-0.067$

Figure 24